ON THE VALUE OF OPTIONS IN CERTAIN CONTRACTS.

To the Editor.

SIR,—Mr. Stephenson's problem on "Options" continuing to excite a good deal of interest, I shall be obliged if you will allow me space for a somewhat full discussion of it.

As stated by Mr. Stephenson the problem is:—To find the single premium for an annuity [of £1] during the remainder of a life (x) after n years, with the condition that the premium is "returnable," without interest, on death or at the option of the purchaser, at any time prior to the commencement of the annuity.

To put this into shape for solution, Mr. S. directs us to consider the single premium, P, as a capital producing in interest Pr per annum; and that this interest, treated as a premium payable at the end of the year, will purchase a certain amount of annuity, to be entered upon in n years. Also, that the capital P being then invested in the purchase of a further amount of annuity, since the whole of the annuity purchased is to be £1, we shall have an equation that will enable us to determine P.

I remark in passing, that the portion of the benefit during the n years here contemplated is slightly different from that in the problem. According to the latter the return is to be P, at the instant of death or withdrawal, the amount of which at the end of the year will be $P(1+r)^{\frac{1}{2}}$. By the other arrangement the amount in the hands of (x) or his representatives, at the same period, would be $P(1+r)$, since, no premium being payable during the year of death or withdrawal, the parties entitled will be in possession of both P and the interest on it for that year. This is a small matter, however.

Mr. Stephenson's solution of the problem, as here modified, is equivalent to the following :—

The present value of the premium Pr, payable at the end of each year for n years, is

$$\frac{Pr(N_x-N_{x+n})}{D_x};$$

and if a be the annuity that this will purchase, we shall have

$$Pr(N_x-N_{x+n})=aN_{x+n}; \ \therefore a=\frac{Pr(N_x-N_{x+n})}{N_{x+n}}.$$

Also, if a' be the annuity that P will purchase at age $x+n$,

$$P=\frac{a'N_{x+n}}{D_{x+n}}; \quad \therefore a'=\frac{PD_{x+n}}{N_{x+n}}.$$

Hence, by condition,

$$\frac{PD_{x+n}+Pr(N_x-N_{x+n})}{N_{x+n}}=a'+a=1;$$

which gives

$$P=\frac{N_{x+n}}{D_{x+n}+r(N_x-N_{x+n})} \quad . \ . \ . \ . \ . \ (1).$$

Now there is a gross paralogism in this process : the very first step is vicious. $Pr(N_x-N_{x+n})\div D_x$ is the value of the premium Pr when it is determinable (during the first n years) by the death of (x) only; but it is

not the value of the same premium when it is determinable *also* at the pleasure or the caprice of (x) by the withdrawal of P.* It is as representing this last-named value that it is employed in the foregoing process, which is therefore vitiated throughout. And the value of P here found is that corresponding to the legitimate signification of the expression referred to; and consequently no provision is made in it for the option of withdrawal.

I shall show this otherwise. Mr. Stephenson's benefit, *divested of the option clause*, and supposing, as above, that the amount of the return at the end of the year of death is $P(1+r)$, consists of, first, an annuity of £1, deferred n years, whose present value is $\frac{N_{x+n}}{D_x}$; and secondly, a temporary assurance for n years of $P(1+r)$, whose present value is $\frac{P(1+r)(M_x-M_{x+n})}{D_x}$. Hence

$$PD_x=N_{x+n}+P(1+r)(M_x-M_{x+n});$$

solution of which gives

$$P=\frac{N_{x+n}}{D_x-(1+r)(M_x-M_{x+n})} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Now (1) and (2) are identical in value: each of them gives, for $x=50$ and $n=10$ (Carlisle, 4 per cent.), $P=6{\cdot}30295$. And they will become identical in symbols also if in (2) we substitute for M_x and M_{x+n} their values as given by the relation $M_x=D_x-(1-v)N_{x-1}=D_x-(1-v)(D_x+N_x)$. It is thus abundantly shown (although it looks like trifling to offer formal demonstration of so plain a matter) that a method which ignores the probability of withdrawal does not, and cannot, assign the true value of a benefit which depends partly on that probability.

If the value of P when returnable *at the moment of death* is required (and still, of course, without the option of withdrawal), we have merely to write $(1+r)^{\frac{1}{2}}$ for $1+r$ in (2), which thus becomes

$$P=\frac{N_{x+n}}{D_x-(1+r)^{\frac{1}{2}}(M_x-M_{x+n})} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

And this gives, for the same values of x and n as before, $P=6{\cdot}2828$. Or, finally, by omitting from (3) the factor $(1+r)^{\frac{1}{2}}$ we get for the value of P, when returnable at the end of the year of death, $6{\cdot}26313$.

The benefit described in the problem, however, is altogether delusive. It seems to have been devised with the object of enabling the purchaser to made a provision for advancing years, while retaining at the same time a certain amount of control over his capital. If so the object can be attained more simply and more efficaciously otherwise. Let the intending purchaser retain his capital in his own hands, and, when his necessities so require, let him invest its improved amount in an immediate annuity, and he will find that he has done just as well as he would have done by the arrangement in the problem, while he has retained his control over both the capital and its interest.

To prove this:—Let the capital as before be P. Its amount in n years will be $P(1+r)^n$; and, if this sum be invested in an annuity of £1, we shall have

* This is very clearly pointed out by Mr. Younger at p. 55 *ante*.

$$P(1+r)^n=\frac{N_{x+n}}{D_{x+n}};$$

whence,
$$P=\frac{N_{x+n}}{(1+r)^n D_{x+n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4);$$

For $x=50$ and $x+n=60$, (1) and (2) give, as we have seen, $P=6{\cdot}30295$; while (4) gives $P=6{\cdot}52820$. This slight increase in the value of P is the equivalent for the advantage of being relieved from the risk of losing the interest in the event of death during the n years.

For the proper solution of Mr. Stephenson's problem it is (or ought to be) obvious, that we require to know the law that governs the withdrawals as well as that which governs the mortality, the influence upon the result exercised by the one of these elements being entirely analogous to that exercised by the other. In attempting a solution of the problem therefore, as a matter of scientific interest, it is necessary to assume a law of withdrawal. This is done by Mr. Younger in the solution given by him in the last number of the *Journal*, pp. 55 to 58; and in that which I am about to offer I shall employ the same law. Of Mr. Younger's solution I shall now only say, that I consider the analysis he employs as of a much more refined character than the circumstances of the case require.

If λ_x denote the number of policies at age x, the numbers of these remaining in force at the end of 1, 2, . . . $t-1$ years, will be denoted by λ_{x+1}, λ_{x+2}, . . . λ_{x+t-1}, respectively. Now, let the law of withdrawal be, that the number of withdrawals in each year is the fraction k of the number of policies in force at the beginning of the year; so shall the withdrawals during the 1st, 2nd . . . tth years be severally denoted by $k\lambda_x$, $k\lambda_{x+1}$ . . . $k\lambda_{x+t-1}$. Let also the withdrawals of each year, and in like manner the deaths, be uniformly distributed over the year.

The decrement of the tth year, if there were no withdrawals, would be the product of λ_{x+t-1} by the probability of $(x+t-1)$ dying in a year. The withdrawals during the year are $k\lambda_{x+t-1}$, at equal intervals therein, which is equivalent, as regards the operation of mortality, to the whole taking place in the middle of the year; and this again is equivalent, in the same sense, to one-half taking place at the commencement of the year, and the other half at the end. Hence the number exposed to the risk of mortality during the whole of the tth year will be denoted by $\lambda_{x+t-1}-\frac{1}{2}k\lambda_{x+t-1}$, or $\lambda_{x+t-1}(1-\frac{1}{2}k)$, which gives for the portion of the decrement due to mortality $\lambda_{x+t-1}(1-p_{x+t-1})(1-\frac{1}{2}k)$; and adding to this the portion due to withdrawals, $k\lambda_{x+t-1}$, we get for the total decrement of the tth year,

$$\lambda_{x+t-1}\{(1-p_{x+t-1})(1-\tfrac{1}{2}k)+k\}.$$

Hence,

$$\lambda_{x+t}=\lambda_{x+t-1}-\lambda_{x+t-1}\{(1-p_{x+t-1})(1-\tfrac{1}{2}k)+k\}$$
$$=\lambda_{x+t-1}\{(1-\tfrac{1}{2}k)p_{x+t-1}-\tfrac{1}{2}k\}.$$

And if we denote by ϖ_{x+t-1} the probability that a policy in force at the beginning of the tth year will not be extinguished that year, but will remain in force at the end, we shall have

$$\varpi_{x+t-1}=\frac{\lambda_{x+t}}{\lambda_{x+t-1}}=(1-\tfrac{1}{2}k)p_{x+t-1}-\tfrac{1}{2}k.$$

From the nature of this function we shall obviously have

$$\lambda_{x+t} = \lambda_x . \varpi_x . \varpi_{x+1} \ . \ . \ . \ \varpi_{x+t-1};$$

and hence, with any assumed value of λ_x as a radix, we shall be able to form with facility a table of the number of policies in force at the end of each successive year, up to age $x+n$, at which the power of withdrawal ceases. The number then in force being subject to diminution by mortality only, the portion of the table from this point will be identical with the corresponding portion of the mortality table if the radix λ_x be so assumed that λ_{x+n} shall equal l_{x+n}. This is a matter easily arranged.

Mr. Younger, in his example, employs the Carlisle mortality, and assumes $k=\frac{1}{20}=\cdot 05$. I have formed the following table on the same hypotheses, to facilitate the comparison of my result with his. It coincides with the Carlisle table at 60, the age at which, in Mr. Younger's example, the power of withdrawal ceases and the annuity is entered upon.

Policy Table.

x.	λ_x.	Deaths and Withdrawals.	Deaths,	Withdrawals.
50	7380·54	465·58	96·56	369·02
51	6914·96	442·10	96·36	345·74
52	6472·86	419·56	95·93	323·63
53	6053·30	397·97	95·31	302·66
54	5655·33	375·94	93·17	282·77
55	5279·39	356·22	92·26	263·96
56	4923·17	337·35	91·20	246·15
57	4585·82	322·72	93·43	229·29
58	4263·10	313·76	100·61	213·15
59	3949·34	306·34	108·87	197·47
60	3643·00	122·00	122·00	

This table, after what precedes, stands in need of no explanation. The last two columns, which show the deaths and withdrawals separately, have been added to afford the means of testing the accordance of the table with the laws used in its formation.

The most convenient way of adapting for use the data with which we are now furnished, will be to form by means of them a commutation table. This I have done as follows:—

Commutation Table.

x.	D'_x.	N'_x.	M'_x.
50	1038·536	9385·130	637·625
51	935·600	8449·530	574·632
52	842·097	7607·433	517·116
53	757·225	6850·208	464·632
54	680·233	6169·975	416·763
55	610·591	5559·384	373·284
56	547·492	5011·892	333·670
57	490·362	4521·530	297·597
58	438·321	4083·209	264·415
59	390·443	3692·766	233·396
60	346·305	3346·461	204·275

The rate of interest involved is 4 per cent., and the table coincides with Mr. Jones's Table XIII. (p. 295), at age 60. To distinguish values affected by withdrawal, I accent the letters designating the columns in which they are found. It is, of course, indifferent whether the values belonging to age 60 be accented or not.

The table is adapted to the treatment of any case in which x is not less than 50 and the age at which the annuity is to be entered upon is 60. In Mr. Younger's example x is 50; and the formula of solution is evidently (3), which for these values of x and $x+n$ becomes

$$P=\frac{{}^{\backprime}N'_{60}}{D'_{50}-(1\cdot04)^{\frac{1}{2}}(M'_{50}-M'_{60})}.$$

The arithmetical operation is as follows:—

M'_{50}	637·625		
M'_{60}	204·275		
	433·350	log	2·636839
	$(1\cdot04)^{\frac{1}{2}}$	„	0·008517
	441·933	„	2·645356
D'_{50}	1038·536		
	596·603	„	2·775685
N'_{60}	3346·46	„	3·524586
P	5·60920	„	0·748901

The value here found for P, 5·60920, is less than that given by (3), 6·2828, which is the value when there are to be no withdrawals. Mr. Younger, however, finds for it 9·4157, which is no less than 50 per cent. greater than that given by (3), as above; and he intimates his opinion that this is what ought to be. It appears to me likely that the opinion thus indicated is merely an inference from his numerical result, since a very little consideration suffices to show that the value here ought to be less than that given by (3), and not greater. Every extinction of a policy previous to the age at which the annuity is to be entered upon (and for some years after) is a decision of the risk on the policy extinguished, in favour of the Office; and hence the introduction of a condition that will increase the number of such extinctions requires and necessitates, for the preservation of the equitable character of the transaction, a diminution and not an increase of the premium. The power to withdraw is thus not a *privilege* accorded to the policy-holder, and for which he ought to be made to pay, but a *concession* made by him, in consenting to abandon his claim to the annuity in consideration of receiving back only the premium he has paid. This being so, then, it follows that Mr. Younger must have fallen into error in his solution of the problem. That he has done so I proceed to show.

We have seen that the expression (3), which, if applied by means of the ordinary table, gives the value of P when there are to be no withdrawals, gives the value also, if applied by means of *a suitable table* when there are to be withdrawals. The expression is

$$P=\frac{N_{x+n}}{D_x-(1+r)^{\frac{1}{2}}M_{x|n}};$$

and it becomes, on division of numerator and denominator by D_x,

$$P=\frac{1}{1-\frac{(1+r)^{\frac{1}{2}}M_{x|n}}{D_x}}\cdot\frac{N_{x+n}}{D_x};$$

or, writing Q for $\frac{(1+r)^{\frac{1}{2}}M_{x|n}}{D_x}$, which is the value of a return of £1,

$$P=\frac{1}{1-Q}\cdot\frac{N_{x+n}}{D_x}.$$

This is Mr. Younger's expression, which, being thus identical with mine, ought, if properly applied—if the proper values of the elements composing it be made use of—to give the same numerical result. Mr. Younger, by an analytical process, which I think I am warranted in calling unnecessarily refined, determines an expression for Q which enables him to assign ·42556 as its value in the case in hand, which value differs but little from that given by my table—namely, ·425534. It is in the remaining element of the expression then—the annuity value—that the principal source of the discrepancy must be sought. Accordingly we find that Mr. Younger here uses the ordinary deferred annuity value; and in so doing—ignoring one of the contingencies on which, during the first 10 years, the value of the annuity depends—he, singular to say, commits an error precisely similar in character to that which he points out as vitiating Mr. Stephenson's investigation. The effect is, that the value thus assigned to P is one that will provide an annuity not only for those who neither die nor withdraw during the 10 years, but also for such of the latter class as shall survive the term!

Using the annuity value derived from my table, and Mr. Younger's value of Q (in which there is probably some arithmetical error), his formula gives for P, 5·60944.* Using also my value of Q, it gives, of course, 5·60920.

I must apologise for having occupied so much space. I hope, however, it may be found that something has been done towards the elucidation of the various points of interest that have arisen.

I am, Sir,

Yours most obediently,

London, 11th May, 1866. P. GRAY.

ON MR. YOUNGER'S LETTER, AND ON THE GENERAL SOLUTION OF PROBLEMS INVOLVING DISTINCT CONTINGENCIES.

To the Editor.

SIR,—After the non-success of my endeavour to convince Mr. Stephenson of the failure of his attempt to solve a new problem in life contingencies, I declared my intention of retiring from the contest; as I felt satisfied that a continuance of the discussion with an opponent who (in the face of the evidence I had adduced) still adhered to his notion that he had succeeded in determining the *value* of the "option of withdrawal," was not likely to lead to any useful result. The subject, however, having been taken up by Mr. Younger, in an able letter published in your last Number, wherein that gentleman (after endeavouring to point out the source of Mr. Stephenson's

* See Mr. Younger's letter, p. 118.—ED. *J. I. A.*